

Vortex states in binary mixture of Bose-Einstein condensates

S. T. Chui^a, V. N. Ryzhov^b and E. E. Tareyeva^b

a: Bartol Research Institute, University of Delaware, Newark, DE 19716

b: Institute for High Pressure Physics, Russian Academy of Sciences, 142 092 Troitsk, Moscow region, Russia

The vortex configurations in the Bose-Einstein condensate of the mixture of two different spin states $|F = 1, m_f = -1\rangle$ and $|2, 1\rangle$ of ^{87}Rb atoms corresponding to the recent experiments by Matthews *et. al.* (Phys. Rev. Lett. **83**, 2498 (1999)) are considered in the framework of the Thomas-Fermi approximation as functions of N_2/N_1 , where N_1 is the number of atoms in the state $|1, -1\rangle$ and N_2 - in the state $|2, 1\rangle$. It is shown that for nonrotating condensates the configuration with the $|1, -1\rangle$ fluid forming the shell about the $|2, 1\rangle$ fluid (configuration "a") has lower energy than the opposite configuration (configuration "b") for all values of N_2/N_1 . When the $|1, -1\rangle$ fluid has net angular momentum and forms an equatorial ring around the resting central condensate $|2, 1\rangle$, the total energy of the system is higher than the ground energy, but the configuration "a" has lower energy than the configuration "b" for all N_2/N_1 . On the other hand, when the $|2\rangle$ fluid has the net angular momentum, for the lowest value of the angular momentum $\hbar l$ ($l = 1$) there is the range of the ratio N_2/N_1 where the configuration "b" has lower energy than the configuration "a". For higher values of the angular momentum the configuration "b" is stable for all values of N_2/N_1 .

The realization of Bose-Einstein Condensation (BEC) in dilute atomic gases offers new opportunities for studying quantum degenerate fluids [1]. These condensates which contain thousands of atoms confined to microscale clouds have similarities to superfluidity and laser, and provide new testing ground for many body physics.

Bose-Einstein condensates are quantum fluids in which many particles have the same quantum state. Intrinsic property of the interacting Bose-Einstein condensate is superfluidity. Bulk superfluids are distinguished from normal fluids by their ability to support dissipationless flow. Superfluidity of ^4He atoms has been widely studied, however, only recently evidence for critical velocity - a key characteristic of superfluids - was observed in a Bose condensate of sodium atoms [2].

Superfluidity is closely related to the existence of stable quantized vortices which are well known for superfluid ^4He . Vortices can be created in superfluid ^4He by cooling a rotating container with helium through the superfluid transition. Quite recently the vortices in one-component Bose-Einstein condensate were generated by simple "stirring" of ^{87}Rb atoms condensate [3]. Vortex states in binary mixture of dilute Bose-Einstein condensates have been created [4] using the method proposed by Williams and Holland [5]. Their new scheme exploits the possibility of simultaneously trapping otherwise identical atoms of ^{87}Rb in two different "hyperfine" spin states $|1\rangle$ and $|2\rangle$. ($|1\rangle$ and $|2\rangle$ denote $|F=1, m_f=-1\rangle$ and $|2, 1\rangle$ spin states of ^{87}Rb atoms respectively). Matthews *et al.* [4] applied a microwave to the condensate and focused a laser beam at various points around its circumference, splitting the atoms into two hyperfine states. The result is a double condensate with one condensate at rest in the center of the cloud and the other in a unit (or multiple) vortex state. Controlling the temporal and spatial dependence of the microwave-induced conversion of $|1\rangle$ into $|2\rangle$ (and vice versa), one can directly create a $|2\rangle$ (or $|1\rangle$) state wave function having a wide variety of shapes out of a $|1\rangle$ (or $|2\rangle$) ground-state wave function [4] or different numbers of atoms in each state.

In [4] two types of vortices have been formed. Matthews *et al.* put the initial condensate into either the $|1\rangle$ or $|2\rangle$ state and then made a vortex in the $|2\rangle$ or $|1\rangle$ state. They considered the time evolution of the vortices to study their dynamics and stability and showed that the dynamics of the $|1\rangle$ state vortices is different from the dynamics of the $|2\rangle$ state vortices due to different scattering lengths.

The simultaneously trapped resting condensates consisting of the ^{87}Rb atoms in the $|1, -1\rangle$ and $|2, 1\rangle$ spin states have been considered experimentally in [6]. In this case the intraspecies and interspecies scattering lengths denoted correspondingly as a_{11}, a_{12}, a_{22} are in the proportion $a_{11} : a_{12} : a_{22} = 1.03 : 1 : 0.97$ with the average of the three being $55(3)\text{\AA}$. It has been observed that the atoms with the larger scattering length a_{11} in the state $|1\rangle$ form a lower-density shell about the atoms with the

smaller scattering length a_{22} [6]. In [4] it has been shown that when the $|1\rangle$ fluid forms an equatorial ring around the resting central fluid this configuration is rather stable. Conversely, the $|2\rangle$ vortex ring sinks in towards the trap center and breaks up.

The aim of this article is to study the relative stability of different vortex configurations as a function of the ratio N_2/N_1 , where N_1 is the number of atoms in the state $|1\rangle$ and N_2 - in the state $|2\rangle$. It is shown that for nonrotating phase-separated condensates the configuration with the $|1\rangle$ fluid forming the shell about the $|2\rangle$ fluid (configuration "a") has lower energy than the opposite configuration (configuration "b") for all values of N_2/N_1 . When the $|1\rangle$ fluid has net angular momentum, the total energy of the system in nonrotating container is higher than the ground energy, but configuration "a" has lower energy than the configuration "b" for all N_2/N_1 . On the other hand, when the $|2\rangle$ fluid has the net angular momentum, for the lowest value of the angular momentum $\hbar l$ ($l=1$) there is the range of the ratio N_2/N_1 where the configuration "b" has lower energy than the configuration "a". For higher values of the angular momentum the configuration "b" is stable for all values of N_2/N_1 . It should be noted that the experiment of Matthews *et al.* [4] corresponds to the case where configuration "b" is unstable ($N_2/N_1 \approx 1, l=1$). In our calculations we use the parameters corresponding to the experiments by Matthews *et al.* [4]: approximately 8×10^5 atom in a spherically symmetric potential with oscillation frequencies of 7.8Hz in the radial and axial directions for both spin states. Let us describe our results in detail.

The modern theoretical description of dilute BEC was originated by the seminal Bogoliubov's 1947 paper where he showed that weak repulsive interaction qualitatively changes the excitation spectra from quadratic free particle form to a linear phonon-like structure. To describe the trapped condensates at $T=0$ one can use the Gross-Pitaevskii (GP) (nonlinear Schrodinger) equation for the condensate wave function [7]. This equation appears as the generalization of the Bogoliubov theory for the inhomogeneous phase. It was widely used to discuss the ground state properties and collective excitations in one-component BEC [8-11].

In order to derive analytic results, some approximations must be used. A commonly used one is the Thomas-Fermi Approximation (TFA), which ignores the kinetic energy terms. It has been shown that in the case of one component condensates the TFA results agree well with the numerical calculations for large particle numbers, except for a small region near the boundary of the condensate [9,10]. In fact, even for small numbers of particles TFA still usually gives qualitatively correct results. In some situations the TFA cannot be used to predict quantitative features of the binary mixture of BEC. For example, the TFA solution considerably underestimate the degree of overlap between the condensates, however, it provides an excellent starting point of study.

Let us first consider the phase separation in binary mixture without rotation.

In the case of a two-species condensate, letting $\psi_i(\mathbf{r})$ ($i = 1, 2$) be the wave function of species i with particle number N_i , we can write two coupled nonlinear Schrodinger (Gross- Pitaevskii) equations as:

$$-\frac{\hbar^2}{2m_1}\nabla^2\psi_1(\mathbf{r}) + \frac{1}{2}m_1\omega_1^2(x^2 + y^2 + \lambda^2 z^2)\psi_1(\mathbf{r}) - \mu_1\psi_1(\mathbf{r}) + G_{11}|\psi_1(\mathbf{r})|^2\psi_1(\mathbf{r}) + G_{12}|\psi_2(\mathbf{r})|^2\psi_1(\mathbf{r}) = 0 \quad (1)$$

$$-\frac{\hbar^2}{2m_2}\nabla^2\psi_2(\mathbf{r}) + \frac{1}{2}m_2\omega_2^2(x^2 + y^2 + \lambda^2 z^2)\psi_2(\mathbf{r}) - \mu_2\psi_2(\mathbf{r}) + G_{22}|\psi_2(\mathbf{r})|^2\psi_2(\mathbf{r}) + G_{12}|\psi_1(\mathbf{r})|^2\psi_2(\mathbf{r}) = 0 \quad (2)$$

Equations (1) and (2) were obtained by minimization of the energy functional of the trapped bosons of masses m_1 and m_2 given by:

$$E(\psi_1, \psi_2) = \int d^3r \left[\frac{\hbar^2}{2m_1} |\nabla\psi_1(\mathbf{r})|^2 + \frac{1}{2}m_1\omega_1^2(x^2 + y^2 + \lambda^2 z^2)|\psi_1(\mathbf{r})|^2 + \frac{\hbar^2}{2m_2} |\nabla\psi_2(\mathbf{r})|^2 + \frac{1}{2}m_2\omega_2^2(x^2 + y^2 + \lambda^2 z^2)|\psi_2(\mathbf{r})|^2 + \frac{G_{11}}{2}|\psi_1(\mathbf{r})|^4 + \frac{G_{22}}{2}|\psi_2(\mathbf{r})|^4 + G_{12}|\psi_1(\mathbf{r})|^2|\psi_2(\mathbf{r})|^2 \right]. \quad (3)$$

The chemical potentials μ_1 and μ_2 are determined by the relations $\int d^3r |\psi_i|^2 = N_i$. The trap potential is approximated by an effective three-dimensional harmonic-oscillator potential well, which is cylindrically symmetric about z axis, λ being the ratio of angular frequencies in the axial direction ω_{zi} to that in the transverse direction $\lambda = \omega_{zi}/\omega_i$. The interaction strengths, G_{11}, G_{22}, G_{12} are determined by the s -wave scattering lengths for binary collisions of like and unlike bosons: $G_{ii} = 4\pi\hbar^2 a_{ii}/m_i$; $G_{12} = 2\pi\hbar^2 a_{12}/m$, where $m^{-1} = m_1^{-1} + m_2^{-1}$.

Let us consider now the phase separation due to interaction between two condensates. In our case of $|1, -1\rangle$ and $|2, 1\rangle$ we have $\frac{1}{2}m_1\omega_1^2 = \frac{1}{2}m_2\omega_2^2$. We simplify the equations by using dimensionless variables. Let us define the length scale $a_\perp = \left(\frac{\hbar}{m_1\omega_1}\right)^{1/2}$, and define the dimensionless variables $\mathbf{r} = a_\perp \mathbf{r}'$, $E = \hbar\omega_1 E'$, $\psi_i(\mathbf{r}) = \sqrt{N_i/a_\perp^3} \psi'_i(\mathbf{r}')$. The wave function $\psi'_i(\mathbf{r}')$ is normalized to 1. In terms of these variables the Gross-Pitaevskii energy functional takes the form:

$$E' = \frac{1}{2} \int d^3r' \left[N_1 |\nabla' \psi'_1|^2 + N_1 (x'^2 + y'^2 + \lambda^2 z'^2) |\psi'_1|^2 + N_2 \beta^2 |\nabla' \psi'_2|^2 + N_2 (x'^2 + y'^2 + \lambda^2 z'^2) |\psi'_2|^2 + \frac{1}{2} N_1 u_1 |\psi'_1|^4 + \frac{1}{2} N_2 u_2 \beta^2 |\psi'_2|^4 + \frac{2\pi a_{12}}{a_\perp} \frac{m_1}{m} N_1 N_2 |\psi'_1|^2 |\psi'_2|^2 \right]. \quad (4)$$

Here $\beta^2 = m_1/m_2 = \omega_2^2/\omega_1^2$ and $u_i = 8\pi a_{ii} N_i/a_\perp$. Eqs. (1) and (2) are rewritten as:

$$-\nabla'^2 \psi'_1 + (x'^2 + y'^2 + \lambda^2 z'^2) \psi'_1 - \mu'_1 \psi'_1 + u_1 |\psi'_1|^2 \psi'_1 + \frac{4\pi a_{12} N_2}{a_\perp} \frac{m_1}{m} |\psi'_2|^2 \psi'_1 = 0; \quad (5)$$

$$-\beta^2 \nabla'^2 \psi'_2 + (x'^2 + y'^2 + \lambda^2 z'^2) \psi'_2 - \mu'_2 \psi'_2 + u_2 \beta^2 |\psi'_2|^2 \psi'_2 + \frac{4\pi a_{12} N_1}{a_\perp} \frac{m_1}{m} |\psi'_1|^2 \psi'_2 = 0; \quad (6)$$

where $\mu'_i = 2\mu_i/\hbar\omega_1$.

In the TFA, Eqs. (4), (5) and (6) can be further simplified by omitting the kinetic energy. For TFA the phase segregated condensates do not overlap, so we can neglect the last terms in Eqs. (4), (5) and (6), obtaining from (5) and (6), in different regions (to be determined later), the simple algebraic equations:

$$|\psi'_1(\mathbf{r}')|^2 = \frac{1}{u_1} (\mu'_1 - (\rho'^2 + \lambda^2 z'^2)); \quad (7)$$

$$|\psi'_2(\mathbf{r}')|^2 = \frac{1}{u_2 \beta^2} (\mu'_2 - (\rho'^2 + \lambda^2 z'^2)). \quad (8)$$

Here $\rho'^2 = x'^2 + y'^2$. From Eqs. (7) and (8) one can see that the condensate density has the ellipsoidal form.

In the case of phase separation, the energy of the system can be written in the form

$$E = E_1 + E_2, \quad (9)$$

where

$$E_1 = \frac{1}{2} \hbar\omega_1 N_1 \left[\mu'_1 - \frac{1}{2} u_1 \int d^3r' |\psi'_1|^4 \right], \quad (10)$$

$$E_2 = \frac{1}{2} \hbar\omega_1 N_2 \left[\mu'_2 - \frac{1}{2} u_2 \beta^2 \int d^3r' |\psi'_2|^4 \right]. \quad (11)$$

In order to obtain Eqs. (10)-(11), Eqs. (7)-(8) have been used.

To investigate the phase separation in the mixture we first suppose that the condensate 1 atoms form an ellipsoidal shell about the condensate 2 atoms (configuration "a"). To determine the position of the boundary between the condensates, we use the condition of thermodynamic equilibrium [12]: the pressures exerted by both condensates must be equal:

$$P_1 = P_2. \quad (12)$$

Pressure is given by [13]:

$$P_i = \frac{G_{ii}}{2} |\psi_i|^4. \quad (13)$$

Condensate 2 has the form of the ellipsoid with long semiaxis q :

$$\rho'^2 + \lambda^2 z'^2 = q^2. \quad (14)$$

From Eqs. (7)-(8) and (12)-(14) one has the equation for q :

$$\mu'_1 - q^2 = \kappa\mu'_2 - \kappa q^2, \quad (15)$$

where $\kappa = \sqrt{(a_{11}m_2)/(a_{22}m_1)}$.

Chemical potentials μ'_1 and μ'_2 can be obtained using the normalization conditions $\int d^3r' |\psi'_1|^2 = \int d^3r' |\psi'_2|^2 = 1$ and are given by:

$$\mu'_1 = \frac{\mu_1^0}{(1 - \frac{5}{2}q'^3 + \frac{3}{2}q'^5)^{2/5}}, \quad (16)$$

$$\mu'_2 = \frac{3}{(\mu'_1)^{3/2}q'^3} \left(\frac{2\beta^2(\mu_2^0)^{5/2}}{15} + \frac{(\mu'_1)^{5/2}q'^5}{5} \right), \quad (17)$$

where $q = \sqrt{\mu'_1}q'$ and

$$\mu_i^0 = \left(\frac{15\lambda u_i}{8\pi} \right)^{2/5}. \quad (18)$$

From equations (16)-(18) one can determine the chemical potentials μ'_1 and μ'_2 and the semiaxis of the phase boundary ellipsoid q as a function of N_1 and N_2 . The energy of the configuration "a" $E_a = E_{a1} + E_{a2}$ is given by:

$$E_{a1} = \frac{1}{2}\hbar\omega_1 N_1 \left\{ \mu'_1 - \frac{15}{4} \frac{(\mu'_1)^{7/2}}{(\mu_1^0)^{5/2}} \times \left[\frac{8}{105} - \left(\frac{q'^3}{3} - \frac{2}{5}q'^5 + \frac{q'^7}{7} \right) \right] \right\}, \quad (19)$$

$$E_{a2} = \frac{1}{2}\hbar\omega_1 N_2 \left\{ \mu'_2 - \frac{15}{4} \frac{(\mu'_1)^{3/2}}{\beta^2(\mu_2^0)^{5/2}} \times \left(\mu'_2 \frac{q'^3}{3} - 2\mu'_2\mu'_1 \frac{q'^5}{5} + \mu_1'^2 \frac{q'^7}{7} \right) \right\}. \quad (20)$$

Let us now consider the opposite case when the condensate $|2\rangle$ atoms form an ellipsoidal shell about the condensate $|1\rangle$ atoms (configuration "b"). In this case Eqs. (15)-(20) can be rewritten in the form:

$$\mu''_1 - q_1^2 = \kappa(\mu''_2 - q_1^2), \quad (21)$$

$$(\mu''_2)^{5/2} = \frac{\beta^2(\mu_2^0)^{5/2}}{1 - \frac{5}{2}q_1'^3 + \frac{3}{2}q_1'^5}, \quad (22)$$

$$\frac{15}{2} \frac{(\mu_2^0)^{3/2}}{(\mu_1^0)^{5/2}} \left(\frac{\mu_1'' q_1'^3}{3} - \frac{\mu_2'' q_1'^5}{5} \right) = 1, \quad (23)$$

$$E_b = E_{b1} + E_{b2},$$

$$E_{b1} = \frac{1}{2}\hbar\omega_1 N_1 \left\{ \mu''_1 - \frac{15}{4} \frac{(\mu_2'')^{3/2}}{(\mu_1^0)^{5/2}} \times \left(\mu_1'' \frac{q_1'^3}{3} - 2\mu_1''\mu_2'' \frac{q_1'^5}{5} + \mu_2''^2 \frac{q_1'^7}{7} \right) \right\}, \quad (24)$$

$$E_{b2} = \frac{1}{2}\hbar\omega_1 N_2 \left\{ \mu''_2 - \frac{15}{4} \frac{(\mu_2'')^{7/2}}{\beta^2(\mu_2^0)^{5/2}} \times \left[\frac{8}{105} - \left(\frac{q_1'^3}{3} - \frac{2}{5}q_1'^5 + \frac{q_1'^7}{7} \right) \right] \right\}. \quad (25)$$

Here μ''_1 and μ''_2 are the chemical potentials in the configuration "b", $q_1 = \sqrt{\mu''_2}q'_1$ is the long semiaxis of the boundary ellipsoid, E_b is the energy of the configuration "b".

To estimate which configuration is stable, one has to compare E_a and E_b .

To evaluate $\Delta E = E_a - E_b$ in general case it is worth first to estimate the energy of the phase boundary which arises due to gradient terms omitted in TFA. The surface energy per unit area, the surface tension, is defined as $\sigma = E_s/S$, where E_s is the surface energy, and S is the interface area. σ may be written in the form [14,15]:

$$\sigma = \frac{\hbar\omega_1}{2\sqrt{2}a_\perp^2} \left(\frac{a_{12}}{\sqrt{a_{11}a_{22}}} - 1 \right)^{1/2} (u_1 u_2 N_1 N_2)^{1/4} \times |\psi'_1| |\psi'_2| (N_1 |\psi'_1|^2 + N_2 |\psi'_2|^2)^{1/2}. \quad (26)$$

Taking into account that the surface area of the ellipsoid with the semiaxis $a_\perp q$ has the form:

$$S = 2\pi a_\perp^2 q^2 \left(1 + \frac{1}{\lambda\sqrt{\lambda^2 - 1}} \log \frac{1}{\lambda - \sqrt{\lambda^2 - 1}} \right), \quad (27)$$

one can estimate the contribution of the surface energy $E_s = \sigma S$ to the total energy of each configuration. To be specific, we will use the parameters corresponding to the experiments of Matthews *et al.* [4] on ^{87}Rb atoms. In this case $m_1 = m_2$, $a_\perp = 3.88 \times 10^{-4} \text{cm}$, $\lambda = 1$, $N = N_1 + N_2 = 0.8 \times 10^6$ atoms. As mentioned in the introduction, $a_{11} : a_{12} : a_{22} = 1.03 : 1 : 0.97$ with the average of the three being $55(3)\text{\AA}$.

In Fig. 1 we show the energies of configurations "a" and "b" (including the surface energy) $E_a/(\hbar\omega_1)$ (solid line) and $E_b/(\hbar\omega_1)$ (dashed line) as functions of $\log_{10}(N_2/N_1)$. One can see that E_a is always lower than E_b . This is consistent with the qualitative assertion and experimental observation that it is energetically favorable for the atoms with the larger scattering length to form a lower-density shell about the atoms with the smaller scattering length [16,17]. It should be noted that the surface energy is much smaller than the interaction energy because the scattering lengths a_{ij} have very close values (see Eq. (26)).

Let us now consider the condensates with nonzero net momentum. For vortex excitation with angular momentum $\hbar l_j$, the condensate wave function is given by

$$\psi_{l_j}(\mathbf{r}) = |\psi_{l_j}(\mathbf{r})| e^{il_j \phi}. \quad (28)$$

After substituting the wave function for the vortex excitation (28) in Eq. (9), the equation describes the ground state of bosons in an effective confinement potential $l_1^2 \hbar^2 / 2m_1 \rho^2 + l_2^2 \hbar^2 / 2m_2 \rho^2 + V_1 + V_2$, where $V_i = m_i \omega_i (\rho^2 + \lambda^2 z^2) / 2$ and $\rho^2 = x^2 + y^2$. So within the TFA the density of the vortex state has the form:

$$|\psi'_1(\mathbf{r}')|^2 = \frac{1}{u_1} \left(\mu'_1(l_1) - (\rho'^2 + \lambda^2 z'^2) - \frac{l_1'^2}{\rho'^2} \right); \quad (29)$$

$$|\psi'_2(\mathbf{r}')|^2 = \frac{1}{u_2 \beta^2} \left(\mu'_2(l_2) - (\rho'^2 + \lambda^2 z'^2) - \frac{\beta^2 l_2'^2}{\rho'^2} \right). \quad (30)$$

The important new qualitative feature of a vortex in the TFA is the appearance of a small hole of radius ξ , $\xi_i^2 \propto l_i^2/\mu_i(l_i)$, but the remainder of the condensate density is essentially unchanged. The fractional change in the chemical potentials caused by the vortex $(\mu'_i(l_i) - \mu'_i)/\mu'_i$ can be shown to be small [8,11], of the order of $1/N^{4/5}$. In calculation of physical quantities containing condensate density it is sufficient to retain the no-vortex density and simply cut off any divergent radial integrals at the appropriate core sizes $\xi_1^2 = l_1^2/\mu'_1$ or $\xi_2^2 = \beta^2 l_2^2/\mu'_2$. Note that using the unperturbed density for calculation of the vortex properties corresponds to the hydrodynamic limit.

In the case of the phase segregated condensate, one finds from Eqs. (29-30) and (10-11) that the energy change due to the presence of the vortices $\Delta E(l_1, l_2)$ has the form:

$$\Delta E = \Delta E_{N_1} + \Delta E_{N_2} = \frac{1}{2} \hbar \omega_1 N_1 \int d_3 r' \frac{l_1^2}{\rho'^2} |\psi'_1|^2 + \frac{1}{2} \hbar \omega_1 N_2 \int d_3 r' \frac{l_2^2 \beta^2}{\rho'^2} |\psi'_2|^2. \quad (31)$$

In the hydrodynamic limit ψ'_i is given by Eqs. (7) and (8).

Let us consider the configuration "a". In the hydrodynamic limit the location of the phase boundary is given by Eq. (14). From (31) one has:

$$\frac{\Delta E_{N_1}^a}{\frac{1}{2} \hbar \omega_1 N_1} = \frac{5l_1^2(\mu'_1)^{3/2}}{(\mu_1^0)^{5/2}} \left\{ \left[\ln \frac{2\mu'_1}{l_1} - \frac{4}{3} \right] - \frac{3}{2} q' \times \right. \\ \left. \times \left[\left(1 - \frac{1}{3} q'^2 \right) \ln \frac{2\mu'_1 q'}{l_1} - \left(1 - \frac{q'^2}{9} \right) \right] \right\} \quad (32)$$

$$\frac{\Delta E_{N_2}^a}{\frac{1}{2} \hbar \omega_1 N_2} = \frac{15l_2^2(\mu'_1)^{1/2} q'}{2(\mu_2^0)^{5/2}} \left[\left(\mu'_2 - \frac{1}{3} \mu'_1 q'^2 \right) \times \right. \\ \left. \times \ln \frac{2\sqrt{\mu'_1 \mu'_2} q'}{l_2 \beta} - \left(\mu'_2 - \frac{\mu'_1 q'^2}{9} \right) \right] \quad (33)$$

The energy for the configuration "b" has the form (the location of the phase boundary is given by Eqs. (21-23)):

$$\frac{\Delta E_{N_2}^b}{\frac{1}{2} \hbar \omega_1 N_2} = \frac{5l_3^2(\mu''_2)^{3/2}}{(\mu_2^0)^{5/2}} \left\{ \left[\ln \frac{2m\mu''_2}{l_3 \beta} - \frac{4}{3} \right] - \frac{3}{2} q'_1 \times \right. \\ \left. \times \left[\left(1 - \frac{1}{3} q_1'^2 \right) \ln \frac{2\mu''_2 q'_1}{l_3 \beta} - \left(1 - \frac{q_1'^2}{9} \right) \right] \right\} \quad (34)$$

$$\frac{\Delta E_{N_1}^b}{\frac{1}{2} \hbar \omega_1 N_1} = \frac{15l_4^2(\mu''_2)^{1/2} q'_1}{2(\mu_1^0)^{5/2}} \left[\left(\mu''_1 - \frac{1}{3} \mu''_2 q_1'^2 \right) \times \right. \\ \left. \times \ln \frac{2\sqrt{\mu''_1 \mu''_2} q'_1}{l_4} - \left(\mu''_1 - \frac{\mu''_2 q_1'^2}{9} \right) \right] \quad (35)$$

Using Eqs.(32-35) one can compare the energies of the configurations for different values of the vortex excitation net angular momenta and different number of state $|1\rangle$ and state $|2\rangle$ atoms. The results of calculations for the case of nonzero net momenta are presented on the figures

2-7. The parameters corresponding to the experiments by Matthews *et al.* [4] were used.

The figures 2-4 show the situation with nonrotating condensate $|2\rangle$ atoms and a vortex of condensate $|1\rangle$ atoms. In Fig.2 we show the rotational parts (32-35) of the energies of configurations "a" and "b" $\Delta E_a/(\hbar \omega_1)$ (solid line) and $\Delta E_b/(\hbar \omega_1)$ (dashed line) as functions of $\log_{10}(N_2/N_1)$ in the case when the condensate of $|1\rangle$ atoms has net angular momentum $l = 1$ (lower curves) and $l = 2$ (upper curves). For the configuration "a" this means the rotation of the outer shell and for the configuration "b" – of the inner part.

In Fig. 3 the corresponding total energy is shown, and Fig. 4 shows the difference between E_a and E_b for this case. One can see that E_a is always lower than E_b .

Let us consider now the case when the condensate $|2\rangle$ atoms has the net angular momentum $l = 1$ or $l = 2$. This case corresponds to the rotation of the outer shell for the configuration "b" and of the interior part of the condensate for the configuration "a". The rotational part is shown on the Fig. 5, the total energy – on Fig. 6. The difference between the total energies E_a and E_b shown in Fig. 7 presents the most interesting of our results. In the $l = 2$ case the "b" configuration with vortex ring of condensate $|2\rangle$ atoms in the outer shell is stable in the whole range of the relative concentrations N_1 and N_2 while in the case of $l = 1$ vortex of condensate $|2\rangle$ atoms this configuration is stable only in the regions of low N_2/N_1 or N_1/N_2 concentrations. When the values N_1 and N_2 approach one another the condensate $|2\rangle$ atoms ring sinks in toward the trap center.

These results are in agreement with the experiment [4] at approximately equal values of N_i : when the condensate $|1\rangle$ has a net angular momentum, it forms an equatorial ring around the central condensate $|2\rangle$. Conversely, a condensate $|2\rangle$ vortex forms a ring that tends to contract down into the condensate $|1\rangle$.

It should be noted that the similar conclusions have been made in Ref. [18]. Using the dynamic stability analysis of the Gross-Pitaevskii equations, the authors of Ref. [18] have investigated the instability mechanism of the configuration "b" and concluded that stabilization of this configuration can be attained by the controlling the relative population of both species.

To summarize, we have shown that for nonrotating phase-separated condensates the configuration with the $|1\rangle$ fluid forming the shell about the $|2\rangle$ fluid (configuration "a") has lower energy than the opposite configuration (configuration "b") for all values of N_2/N_1 . When the $|1\rangle$ fluid has net angular momentum, the total energy of the system is higher than the ground energy, but the configuration "a" has lower energy than the configuration "b" for all N_2/N_1 . On the other hand, when the $|2\rangle$ fluid has the net angular momentum, for the lowest value of the angular momentum $\hbar l$ ($l = 1$) there is the range of the ratio N_2/N_1 where the configuration "b" has lower energy than the configuration "a". For higher values of the angular momentum the configuration "b"

is stable for all values of N_2/N_1 .

This work was supported in part by NATO Grant No.PST.CLG.976038. V.N.R and E.E.T acknowledge the financial support from the Russian Science Foundation through the Grant No.98-02-16805. STC was partly supported by NASA under grant No. CRG8-1427. The authors are grateful to V.M. Perez-Garcia and J.J. Garcia-Ripoll for drawing their attention to Ref. [18].

-
- [1] M.N. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, *Science* **269**, 198 (1995).
 - [2] C. Raman *et al.*, *Phys. Rev. Lett.* **83**, 2502 (1999).
 - [3] K. W. Madison, F. Chevy, W. Wohlleben and J. Dalibard, *Phys. Rev. Lett.* **84**, 806 (2000).
 - [4] M. R. Matthews *et al.*, *Phys. Rev. Lett.* **83**, 2498 (1999).
 - [5] J. E. Williams and M. J. Holland, *Nature* **401**, 568 (1999).
 - [6] D.S. Hall, M.R. Matthews, J.R. Ensher, C.E. Wieman, and E.A. Cornell, cond-mat/9804138; E.A. Cornell, D.S. Hall, M.R. Matthews, and C.E. Wieman, cond-mat/9808105.
 - [7] L.P. Pitaevskii, *Sov. Phys. JETP* **13**, 451 (1961); E.P. Gross, *Nuovo Cimento* **20**, 454 (1961); J. Math. Phys. **4**, 195 (1963).
 - [8] A.L. Fetter, cond-mat/9808070; cond-mat/9811316.
 - [9] G. Baym and C.J. Pethick, *Phys. Rev. Lett.* **76**, 6 (1996).
 - [10] F. Dalfovo and S. Stringari, *Phys. Rev. A* **53**, 2477 (1996).
 - [11] S. Sinha, *Phys. Rev. A* **55**, 4325 (1997).
 - [12] L.D. Landau and E.M. Lifshits. *Statistical Physics. Part I*. 3d edition. Moscow, 1976.
 - [13] L.P. Pitaevskii, *Physics - Uspekhi* **41**, 569 (1998).
 - [14] E. Timmermans, *Phys. Rev. Lett.* **81**, 5718 (1998).
 - [15] P. Ao and S.-T. Chui, *Phys. Rev. A* **58**, 4836 (1998).
 - [16] D.S. Hall, M.R. Matthews, J.R. Ensher, C.E. Wieman, and E.A. Cornell, cond-mat/9804138; E.A. Cornell, D.S. Hall, M.R. Matthews, and C.E. Wieman, cond-mat/9808105.
 - [17] H. Pu and N.P. Bigelow, *Phys. Rev. Lett.* **80**, 1130 (1998).
 - [18] V.M. Perez-Garcia and J.J. Garcia-Ripoll, cond-mat/9912308; *Phys. Rev. Lett.* **84**, 4264 (2000).

FIG. 1. The total energies of configurations "a" and "b" $E_a/(\hbar\omega_1 N)$ (solid line) and $E_b/(\hbar\omega_1 N)$ (dashed line) for non rotating condensate as functions of $\log_{10}(N_2/N_1)$.

FIG. 2. The rotational energies of configurations "a" and "b" $\Delta E_a/(\hbar\omega_1)$ (solid line) and $\Delta E_b/(\hbar\omega_1)$ (dashed line) as functions of $\log_{10}(N_2/N_1)$ in the case when the condensate of $|1\rangle$ atoms has net angular momentum $l = 1$ (lower curves) and $l = 2$ (upper curves).

FIG. 3. The total energies of configurations "a" and "b" $E_a/(\hbar\omega_1)$ (solid line) and $E_b/(\hbar\omega_1)$ (dashed line) as functions of $\log_{10}(N_2/N_1)$ in the case when the condensate of $|1\rangle$ atoms has net angular momentum $l = 1$ (lower curves) and $l = 2$ (upper curves).

FIG. 4. The difference of total energies for configurations "a" and "b" $(E_a - E_b)/(\hbar\omega_1)$ as a function of $\log_{10}(N_2/N_1)$ in the case when the condensate of $|1\rangle$ atoms has net angular momentum $l = 1$ (solid line) and $l = 2$ (dashed line).

FIG. 5. The rotational energies of configurations "a" and "b" $\Delta E_a/(\hbar\omega_1)$ (solid line) and $\Delta E_b/(\hbar\omega_1)$ (dashed line) as functions of $\log_{10}(N_2/N_1)$ in the case when the condensate of $|2\rangle$ atoms has net angular momentum $l = 1$ (lower curves) and $l = 2$ (upper curves).

FIG. 6. The total energies of configurations "a" and "b" $E_a/(\hbar\omega_1)$ (solid line) and $E_b/(\hbar\omega_1)$ (dashed line) as functions of $\log_{10}(N_2/N_1)$ in the case when the condensate of $|2\rangle$ atoms has net angular momentum $l = 1$ (lower curves) and $l = 2$ (upper curves).

FIG. 7. The difference of total energies for configurations "a" and "b" $(E_a - E_b)/(\hbar\omega_1)$ as a function of $\log_{10}(N_2/N_1)$ in the case when the condensate of $|2\rangle$ atoms has net angular momentum $l = 1$ (solid line) and $l = 2$ (dashed line).













